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Level 3 Calculus, 2021 v1

91577 Apply the algebra of complex numbers in solving problems

Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

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TOTAL

QUESTION ONE

(a) If $u = k\text{cis}\left(\frac{\pi}{3}\right)$ and $v = k^3\text{cis}\left(\frac{\pi}{4}\right)$, find $\frac{u}{v}$ in polar form.

(b) Let $x = a + 2i$ and $y = 3 - 7i$, where a is a real number.

Find the value of a if $\frac{xy}{2} = 1 + 17i$.

(c) Solve the equation $\sqrt{x} + \sqrt{x-1} = p$ for x in terms of p.

ASSESSOR'S USE ONLY

QUESTION TWO

(a) Convert $4-4i$ to polar form.

(b) The number $z=a+bi$, where a and b are real numbers, satisfies the equation $\overline{z-zi}=5+3i$.
Find the values of a and b.

(c) Express the polynomial $2x^3-3x^2-8x-3$ in the form $(ax+b)(x+c)(x+d)$, where a, b, c and d are integers.

ASSESSOR'S USE ONLY

(e) The complex number z satisfies $z\overline{z} + \frac{\overline{z}}{z} = 1$ and $\text{Im}(z) \neq 0$.

(i) Prove that $\overline{z} = -z$.

(ii) Hence or otherwise, find the value of z^2.

Assessment Schedule – 2021 v1

Calculus: Apply the algebra of complex numbers in solving problems (91577)

Evidence Statement

Q1	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$\frac{u}{v}=\frac{k\text{cis}\left(\frac{\pi}{3}\right)}{k^3\text{cis}\left(\frac{\pi}{4}\right)}=\frac{1}{k^2}\text{cis}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\frac{1}{k^2}\text{cis}\left(\frac{\pi}{12}\right)$	Correct answer.		
(b)	$\frac{xy}{2}=\frac{(a+2i)(3-7i)}{2}=\frac{3a+14}{2}+\frac{6-7a}{2}i$ Comparing the real parts (or the imaginary parts) gives $a=-4$.	Expressed $\frac{xy}{2}$ in rectangular form in terms of a.	Correct value of a.	
(c)	$\sqrt{x}+\sqrt{x-1}=p$ $\sqrt{x-1}=p-\sqrt{x}$ $\sqrt{x-1}^2=\left(p-\sqrt{x}\right)^2$ $x-1=p^2-2p\sqrt{x}+x$ $2p\sqrt{x}=p^2+1$ $\sqrt{x}=\frac{p^2+1}{2p}$ $x=\left(\frac{p^2+1}{2p}\right)^2$	Partial working with terms containing x on one side (line 5 or 6).	Correct solution.	
(d)	The equation has no real roots if the discriminant of the equation is negative. $\Delta=s^2-4\times1\times(s+1)$ $=s^2-4s-4$ $=(s-2)^2-8$ $=\left(s-2+2\sqrt{2}\right)\left(s-2-2\sqrt{2}\right)$ $\Delta<0$ when $2-2\sqrt{2}<s<2+2\sqrt{2}$ *Accept rounded values –0.8284 and 4.8284.*	Discriminant of the quadratic is found and simplified.	Correct restriction on s.	

Q3	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	Let $z = x + iy$. $\lvert 4z-1\rvert = \lvert z-1\rvert$ $\lvert 4(x+yi)-1\rvert = \lvert x+yi-1\rvert$ $\lvert 4x-1+4yi\rvert = \lvert x-1+yi\rvert$ $\sqrt{(4x-1)^2+(4y)^2} = \sqrt{(x-1)^2+y^2}$ $16x^2-8x+1+16y^2 = x^2-2x+1+y^2$ $15x^2-6x+15y^2=0$ $x^2-\frac{2}{5}x+\frac{1}{25}+y^2=\frac{1}{25}$ $\left(x-\frac{1}{5}\right)^2+y^2=\frac{1}{25}$	Equation equating moduli without absolute value signs (line 5).	Equation for locus with square roots removed (line 6).	Correct Cartesian equation of the locus is obtained.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

Cut Scores

	Not Achieved	Achievement	Achievement with merit	Achievement with Excellence
Score range	0 – 7	8 – 14	15 – 20	21 – 24

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9 1 5 7 8

Level 3 Calculus, 2021 v1

91578 Apply differentiation methods in solving problems

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

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TOTAL

QUESTION ONE

ASSESSOR'S USE ONLY

(a) Differentiate $y=(4x^5+2x^3)^5$.

You do not need to simplify your answer.

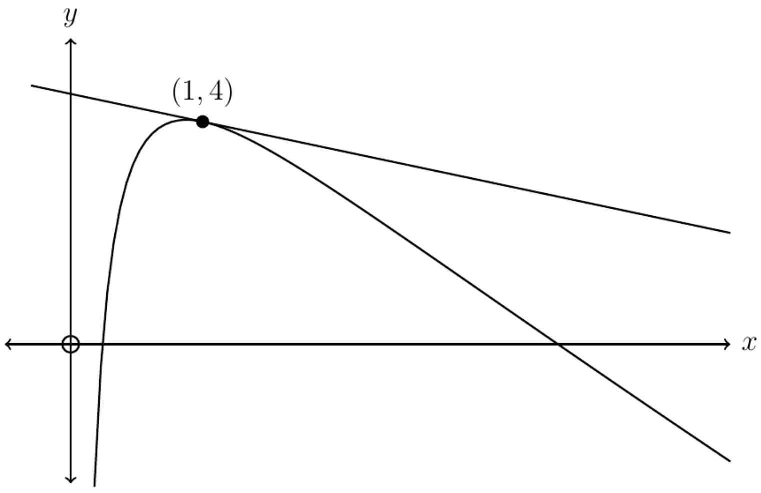
(b) Find the gradient of the tangent to the curve $y = x+\cos 2x-\frac{\pi}{2}$ at the point where $x=\frac{\pi}{2}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

(c) Show that $f(x)=\ln x+\frac{1}{4x}-x$ is always decreasing.

You must use calculus and show any derivatives that you need to find when solving this problem.

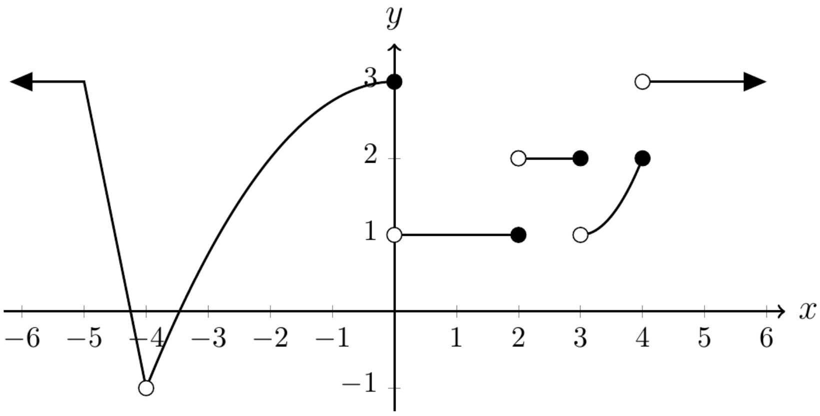
(d) Find the equation of the normal to the curve $y = 10 - x - 3\sqrt{x} - \frac{2}{x}$ at the point $(1, 4)$.



You must use calculus and show any derivatives that you need to find when solving this problem.

(c) The diagram below shows the graph of the function $y = f(x)$.

ASSESSOR'S USE ONLY



For the function above:

(i) Find the value(s) for x that meet the following conditions:

1. $f(x)$ is not defined:__________________________________

2. $f'(x) > 0$:__________________________________

3. $f(x)$ is continuous but not differentiable:__________________________

(ii) What is the value of $f(3)$?__________________________________

State clearly if the value does not exist.

(iii) What is the value of $\lim_{x \to -4} f(x)$?__________________________________

State clearly if the value does not exist.

Assessment Schedule – 2021 v1

Calculus: Apply differentiation methods in solving problems (91578)

Evidence Statement

Q1	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$\frac{dy}{dx} = 5\times(4x^5+2x^3)^4\times(20x^4+6x^2)$	Correct derivative.		
(b)	$y = x+\cos 2x - \frac{\pi}{2}$ $\frac{dy}{dx} = 1-2\sin 2x$ At $x=\frac{\pi}{2}$ $\frac{dy}{dx} = 1-2\sin\left(2\times\frac{\pi}{2}\right) = 1-2\sin\pi = 1$	Correct gradient with correct derivative.		
(c)	$f(x) = \ln x + \frac{1}{4x} - x = \ln x + \frac{1}{4}x^{-1} - x$ $f'(x) = \frac{1}{x} - \frac{1}{4}x^{-2} - 1$ $= -\frac{4x^2-4x+1}{4x^2}$ $= -\frac{(2x-1)^2}{4x^2}$ Therefore, $f'(x) < 0$ for all x.	Correct $f'(x)$.	Correct justification.	
(d)	$y = 10 - x - 3\sqrt{x} - \frac{2}{x} = 10 - x - 3x^{\frac{1}{2}} - 2x^{-1}$ $\frac{dy}{dx} = -1 - \frac{3}{2\sqrt{x}} + \frac{2}{x^2}$ At $x=1$ $\frac{dy}{dx} = -1 - \frac{3}{2\sqrt{1}} + \frac{2}{1^2} = -\frac{1}{2}$ This means that gradient of the normal at this point is 2. Therefore, the equation of the normal is $y-4 = 2(x-1)$ which is equivalent to $y = 2x+2$.	Correct gradient of tangent OR Correct gradient of normal.	Correct equation of normal.	

Q2	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$\dfrac{2\cos 2x \times (\cos x + \tan x) - \sin 2x \times (-\sin x + \sec^2 x)}{(\cos x + \tan x)^2}$ ● $\sec^2 x$ can be written as $\dfrac{1}{\cos^2 x}$ instead.	Correct derivative.		
(b)	$\dfrac{dy}{dx} = -e^{1-x} \times x + e^{1-x} \times 1$ $= e^{1-x}(-x+1)$ Solving $\dfrac{dy}{dx} = 0$ gives $e^{1-x}(-x+1) = 0$ $x = 1$	Correct $\dfrac{dy}{dx}$.	Correct x value for the stationary point.	
(c)	(i) 1) -4 2) $-4 < x < 0$ and $3 < x < 4$ 3) $x = -5$ (ii) 2 (iii) -1	TWO out of five answers correct.	THREE out of five answers correct.	
(d)	$V = 0.01e^{0.2t} + 0.49e^{-0.2t}$ $\dfrac{dV}{dt} = 0.002e^{0.2t} - 0.098e^{-0.2t}$ $= 0.002e^{-0.2t}\left(e^{0.4t} - 49\right)$ Solving $\dfrac{dV}{dt} = 0$ gives $0.002e^{-0.2t}\left(e^{0.4t} - 49\right) = 0$ $e^{0.4t} = 49$ $t = \dfrac{\ln(49)}{0.4}$ Substituting this value into V gives $V = 0.01e^{0.2\times\frac{\ln(49)}{0.4}} + 0.49e^{-0.2\times\frac{\ln(49)}{0.4}} = 0.14\ \text{cm}^3$	Correct value for t with correct derivative.	Correct volume. Units are not required.	

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Level 3 Calculus, 2021 v1

91579 Apply integration methods in solving problems

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply integration methods in solving problems.	Apply integration methods, using relational thinking, in solving problems.	Apply integration methods, using extended abstract thinking, in solving problems.

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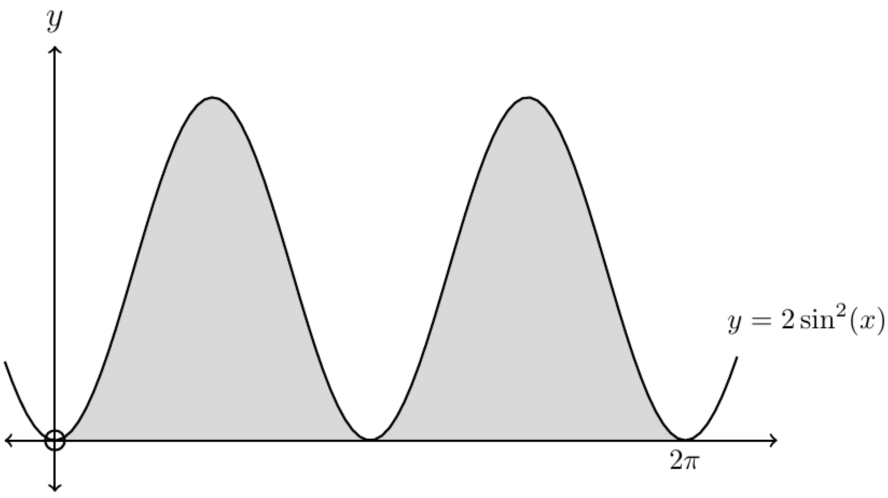
TOTAL

(d) The diagram below shows the graph of the function $y = 2\sin^2(x)$.



Find the area of the shaded region.

You must use calculus and show the results of any integration needed to solve the problem.

ASSESSOR'S USE ONLY

ASSESSOR'S USE ONLY

(e) An object's acceleration can be modelled by the function $a(t) = \dfrac{1}{(t+2)\sqrt{t+2}}$, where $t \geq 0$.

a is the acceleration of the object in ms^{-2} and
t is the time in seconds since the start of the object's motion.

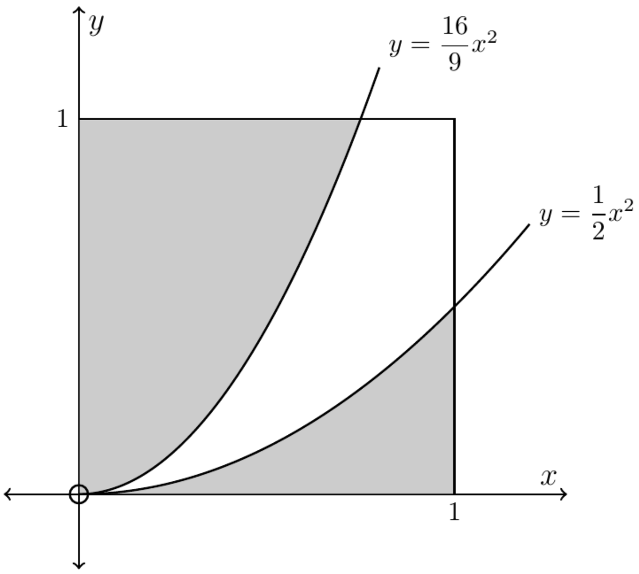
The object's velocity is 6 ms^{-1} when $t = 2$.

How far did the object move in the 3rd second of its motion?

You must use calculus and show the results of any integration needed to solve the problem.

(c) The design of a square tile is to be shaded as shown below.

The graphs of the curved lines can be modelled by the functions $y = \frac{16}{9}x^2$ and $y = \frac{1}{2}x^2$.



Use integration to find the total area of the shaded regions of the tile.

You must use calculus and show the results of any integration needed to solve the problem.

ASSESSOR'S USE ONLY

Assessment Schedule – 2021 v1
Calculus: Apply integration methods in solving problems (91579)
Evidence Statement

Q1	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$\frac{e^{2x}}{2}+e^x+c$	Correct solution.		
(b)	$\frac{0.6}{3}\left(2.4+0.6+4\times(3.9+5.2+1.5)+2\times(4.7+2.3)\right)$ $=11.88$	Correct solution.		
(c)	$\int_0^k x\times\sqrt[3]{x}\ dx=\int_0^k x^{\frac{4}{3}}\ dx$ $=\left[\frac{3}{7}x^{\frac{7}{3}}\right]_0^k$ $=\frac{3}{7}k^{\frac{7}{3}}$ $\frac{3}{7}k^{\frac{7}{3}}=\frac{3}{7}$ when $k=1$.	Correct integration of $x\sqrt[3]{x}$	Correct solution with correct integration of $x\sqrt[3]{x}$.	
(d)	Using the double angle formula $\sin^2(x)=\frac{1}{2}\left(1-\cos(2x)\right)$ $2\sin^2(x)=1-\cos(2x)$ Area $=\int_0^{2\pi}2\sin^2(x)\ dx$ $=4\int_0^{\pi/2}2\sin^2(x)\ dx$ $=4\int_0^{\pi/2}1-\cos(2x)\ dx$ $=4\left[x-\frac{1}{2}\sin(2x)\right]_0^{\pi/2}$ $=4\left(\left(\frac{\pi}{2}-\frac{1}{2}\sin(\pi)\right)-\left(0-\frac{1}{2}\sin(0)\right)\right)$ $=2\pi$ • Accept 6.28.	Correct integration.	Correct solution with correct integration.	

Q2	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$\sqrt{x}=\frac{3}{2}-\frac{x}{2}$ $x=\left(\frac{1}{2}(3-x)\right)^2$ $4x=x^2-6x+9$ $x^2-10x+9=0$ $(x-1)(x-9)=0$ $x=1,\ 9$ The straight line and the square root function intersect at $x = 1$. The square root function and the parabola intersect at $x = 0$. Also, the linear function and the parabola intersect at $x = 3$. $\text{Area } 1=\int_0^1\sqrt{x}\ dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1=\frac{2}{3}$ $\text{Area } 2=\int_1^3\left(\frac{3}{2}-\frac{x}{2}\right)dx=\left[\frac{3x}{2}-\frac{x^2}{4}\right]_1^3$ $=\left(\frac{3(3)}{2}-\frac{(3)^2}{4}\right)-\left(\frac{3(1)}{2}-\frac{(1)^2}{4}\right)$ $=1$ $\text{Area } 3=-\int_0^3\left(x^2-3x\right)dx=-\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^3$ $=-\left(\frac{(3)^3}{3}-\frac{3(3)^2}{2}\right)$ $=\frac{9}{2}$ Therefore, the area of the shaded region is $\frac{2}{3}+1+\frac{9}{2}=\frac{37}{6}$.	One of the integrals is correct.	Two of the integrals are correct.	Correct answer with correct integrals. Accept 6.17.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of integration techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

Q3	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$$\frac{dQ}{dt}=\frac{1}{1.2}(0.024-Q)$$ $$\int\frac{1}{0.024-Q}\,dq=\int\frac{1}{1.2}\,dt$$ $$-\ln\lvert 0.024-Q\rvert=\frac{t}{1.2}+c$$ Since $Q=0$ when $t=0$, $c=-\ln\lvert 0.024\rvert$. $$-\ln\lvert 0.024-Q\rvert=\frac{t}{1.2}-\ln\lvert 0.024\rvert$$ $$\ln\lvert 0.024-Q\rvert-\ln\lvert 0.024\rvert=-\frac{t}{1.2}$$ $$\ln\left\lvert\frac{0.024-Q}{0.024}\right\rvert=-\frac{t}{1.2}$$ $$\frac{0.024-Q}{0.024}=e^{-\frac{t}{1.2}}$$ $$Q=0.024\left(1-e^{-\frac{t}{1.2}}\right)$$ From the equation obtained, the amount of charge in the capacitor has a limiting value of 0.024 Farads $$0.024\left(1-e^{-\frac{t}{1.2}}\right)=0.024\times\frac{1}{2}$$ $$e^{-\frac{t}{1.2}}=\frac{1}{2}$$ $$-\frac{t}{1.2}=\ln\left(\frac{1}{2}\right)$$ $$t=-1.2\ln\left(\frac{1}{2}\right)\approx 0.83$$ The capacitor will be half full after 0.83 seconds.	Correct integration.	Obtained the correct function for Q of t.	Correct answer with correct integration.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of integration techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t