3

9 1 5 7 7

Level 3 Calculus, 2020 v1

91577 Apply the algebra of complex numbers in solving problems

Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

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TOTAL

QUESTION ONE

(a) $(x+1)$ is a factor of the polynomial $2x^3 - x^2 + kx - 2$.
What is the value of the constant k?

(b) Solve the equation $x^2 - 2\sqrt{3}x + 4 = 0$ for x.
Give your solutions in the form $\sqrt{a} \pm bi$, where a and b are real numbers.

(c) $(1+i)(i+3x) = 5 + yi$, where x and y are real numbers.
Find the values of x and y.

ASSESSOR'S USE ONLY

(c) $z_1 = 1 + i$ and $z_2 = 1 - \sqrt{3}i$.

Find $\arg(z_1 z_2)$.

(d) What are the square roots of $-2 - 2\sqrt{3}i$ in polar form?

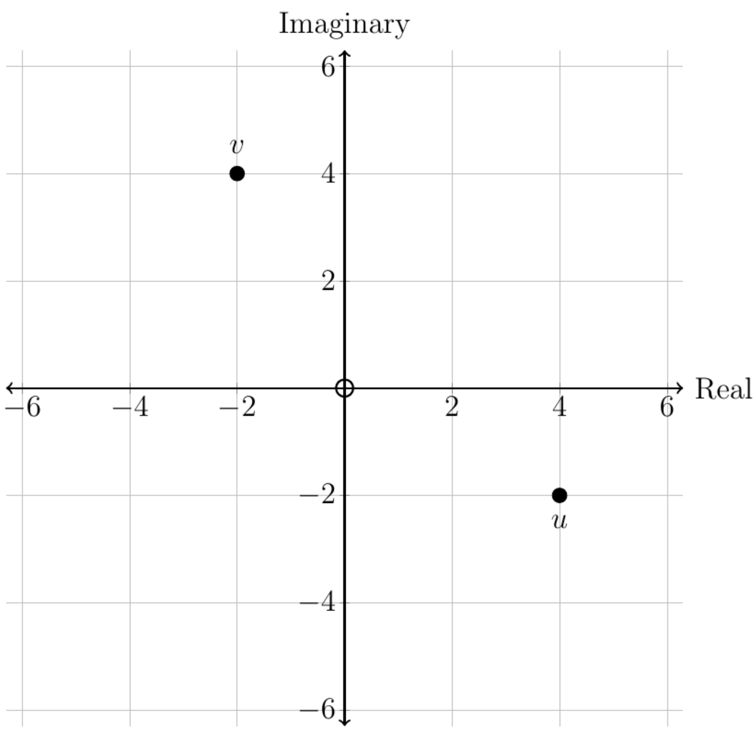
(e) Solve the equation $\sqrt{x-3p}+\sqrt{x+p}=\sqrt{x+3p}$, where p is a positive real number.

ASSESSOR'S USE ONLY

QUESTION THREE

(a) Complex numbers u and v are represented on the Argand diagram below.

If $z = \overline{u + v}$, show z on the Argand diagram.



(b) Prove that for complex numbers z_1 and z_2, $\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2}$.

Q1	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$z+\overline{z}=\frac{1+\sqrt{2}i}{2}+\frac{1-\sqrt{2}i}{2}=\frac{2}{2}=1$ $z\overline{z}=\frac{1+\sqrt{2}i}{2}\times\frac{1-\sqrt{2}i}{2}=\frac{1+2}{4}=\frac{3}{4}$ Therefore $w\overline{w}=\frac{3z+1}{5z-1}\times\overline{\left(\frac{3z+1}{5z-1}\right)}$ $=\frac{3z+1}{5z-1}\times\frac{3\overline{z}+1}{5\overline{z}-1}$ $=\frac{9z\overline{z}+3(z+\overline{z})+1}{25z\overline{z}-5(z+\overline{z})+1}$ $=\frac{9\times\frac{3}{4}+3+1}{25\times\frac{3}{4}-5+1}$ $=\frac{27+16}{75-16}$ $=\frac{43}{59}$	Found the correct values for $z+\overline{z}$ or $z\overline{z}$.	Simplified $w\overline{w}$ by expanding brackets (line 3).	Correct steps taken to show the required equality.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

Q2	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$z^2 = -s^2 + 2s - 1$ $= -(s^2 - 2s + 1)$ $= -(s-1)^2$ $z = \pm\sqrt{-(s-1)^2} = \pm(s-1)i$	Correct answer.		
(b)	$\frac{5}{2-\sqrt{3}} = \frac{5}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$ $= \frac{5\left(2+\sqrt{3}\right)}{4-3}$ $= 10 + 5\sqrt{3}$ Accept $5(2+\sqrt{3})$.	Correct answer.		
(c)	$z_1 z_2 = \left(1+i\right)\left(1-\sqrt{3}i\right)$ $= \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) \times 2\text{cis}\left(-\frac{\pi}{3}\right)$ $= 2\sqrt{2}\text{cis}\left(\frac{3\pi}{12} - \frac{4\pi}{12}\right)$ $= 2\sqrt{2}\text{cis}\left(-\frac{\pi}{12}\right)$ Therefore, $\arg(z_1 z_2) = -\frac{\pi}{12}$. Accept -0.262.	Found $z_1 z_2$ in polar form.	Correct answer.	
(d)	Let $z = \sqrt{-2-2\sqrt{3}i}$. $z^2 = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2}\,\text{cis}\left(-\frac{2\pi}{3} + 2n\pi\right)$ $z^2 = 4\text{cis}\left(-\frac{2\pi}{3} + 2n\pi\right)$ $z = 2\text{cis}\left(-\frac{\pi}{3} + n\pi\right)$ where n is an integer. When $n = 0$, $z = 2\text{cis}\left(-\frac{\pi}{3}\right)$ When $n = 1$, $z = 2\text{cis}\left(\frac{2\pi}{3}\right)$	Correct z written down. AND Correct z^2 in polar form.	Correct solutions.	

Q3	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(d)	By the conjugate root theorem $z = 2 + i$ is also a solution. $\left(z-(2-i)\right)\left(z-(2+i)\right)(z-k)=0$ $\left(z-2+i\right)\left(z-2-i\right)(z-k)=0$ $(z^2-4z+5)(z-k)=0$ $z^3-4z^2+5z-kz^2+4kz-5k=0$ $z^3-(4+k)z^2+(5+4k)z-5k=0$ $k=-1$, therefore, the last solution is $z=-1$.	Stated that $z = 2 + i$ is another solution.	Found the last solution $z = -1$ with correct steps.	
(e)	Let $z = x + iy$. $\lvert z+m-mi\rvert = \lvert z-m+mi\rvert$ $\lvert x+yi+m-mi\rvert = \lvert x+yi-m+mi\rvert$ $\lvert (x+m)+(y-m)i\rvert = \lvert (x-m)+(y+m)i\rvert$ $\sqrt{(x+m)^2+(y-m)^2} = \sqrt{(x-m)^2+(y+m)^2}$ $(x+m)^2+(y-m)^2 = (x-m)^2+(y+m)^2$ $(x+m)^2-(x-m)^2 = (y+m)^2-(y-m)^2$ $2x(2m) = 2y(2m)$ $x = y$ A straight line with a gradient of 1 passing through the origin $(0,0)$.	Equation equating moduli without absolute value signs (line 4).	Equation for locus with squared terms cancelled (line 7).	Correct description of the locus.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

Cut Scores

	Not Achieved	Achievement	Achievement with merit	Achievement with Excellence
Score range	0 – 7	8 – 14	15 – 20	21 – 24

3

9 1 5 7 8

Level 3 Calculus, 2020 v1

91578 Apply differentiation methods in solving problems

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

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Show ALL working.

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TOTAL

QUESTION ONE

ASSESSOR'S USE ONLY

(a) Differentiate $f(x)=\sqrt{2x^3-4x}$.

You do not need to simplify your answer.

(b) For $y=xe^x$, show that $\frac{d^2y}{dx^2}-\frac{dy}{dx}=e^x$.

(e) A curve is defined parametrically by the equations

$$x = a^2(t - \sin t \cos t) + 2a$$

$$y = a^2(\sin t)^2$$

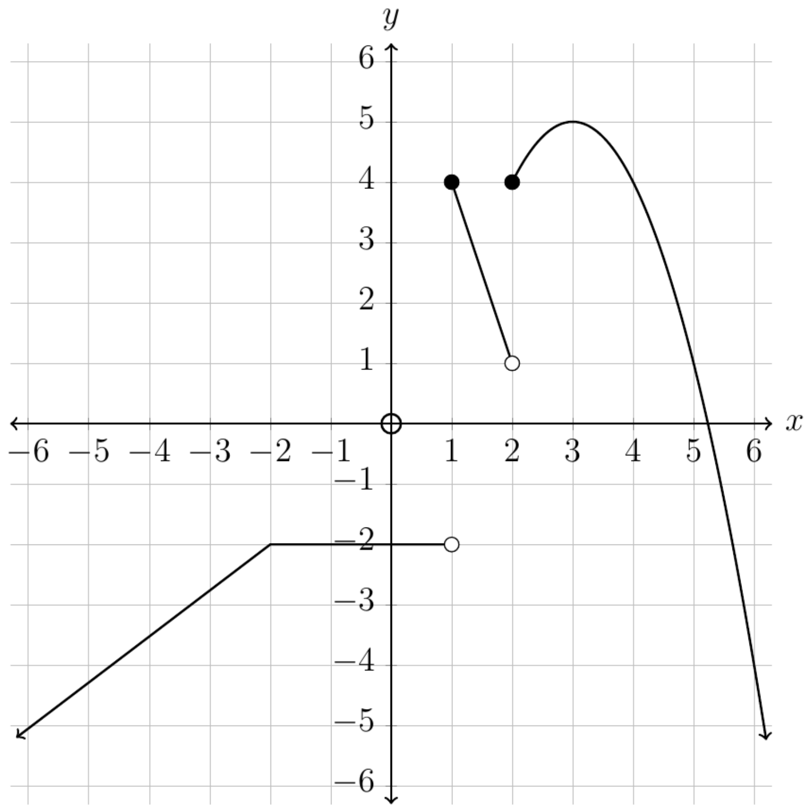
where a is a non-zero constant.

It is given that $\left(1+\left(\frac{dy}{dx}\right)^2\right)y = 4k^2$.

Find a in terms of k.

ASSESSOR'S USE ONLY

(c) The diagram below shows the graph of the function $y = f(x)$.



For the function above:

(i) 1. Find where f is not continuous

2. Find all value(s) of x where $f'(x) > 0$.

3. Find all value(s) of x where $f''(x) < 0$.

(ii) What is the value of $\lim_{x \to 2} f(x)$?

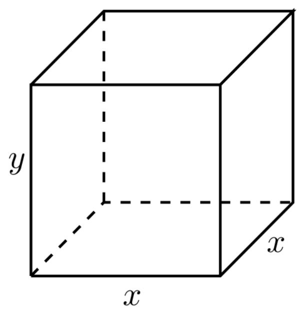
State clearly if the value does not exist.

QUESTION THREE

(a) Differentiate $g(x) = \dfrac{\cos 2x}{x^2 + 1}$.

You do not need to simplify your answer.

(b) The surface area of the cuboid shown below is $2a^2$, where $a > 0$.



What is the value of x that gives the maximum volume of the cuboid?

You must use calculus and show any derivatives that you need to find when solving this problem.

Assessment Schedule – 2020 v1

Calculus: Apply differentiation methods in solving problems (91578)

Evidence Statement

Q1	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$f'(x)=\frac{1}{2}\left(2x^3-4x\right)^{-\frac{1}{2}}\times\left(6x^2-4\right)$	Correct derivative.		
(b)	$\frac{dy}{dx}=e^x+xe^x=(1+x)e^x$ $\frac{d^2y}{dx^2}=e^x+(1+x)e^x$ $=(2+x)e^x$ $\frac{d^2y}{dx^2}-\frac{dy}{dx}=(2+x)e^x-(1+x)e^x$ $=e^x$	Correct proof with correct $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.		
(c)	$y=\ln(x+1)^2=2\ln(x+1)$ $\frac{dy}{dx}=\frac{2}{x+1}$ At the point $(e-1,2)$ $\frac{dy}{dx}=\frac{2}{e-1+1}=\frac{2}{e}$ Therefore, the gradient of the normal is $-\frac{e}{2}$. Accept -1.359.	Correct expression for $\frac{dy}{dx}$.	Correct answer with correct expression for $\frac{dy}{dx}$.	
(d)	$v=\frac{dx}{dt}=3\pi\sin\left(\frac{3\pi}{2}t-\frac{\pi}{2}\right)$ At $t=\frac{2}{3}$ $v=3\pi\sin\left(\frac{3\pi}{2}\times\frac{2}{3}-\frac{\pi}{2}\right)$ $=3\pi\sin\left(\frac{\pi}{2}\right)$ $=3\pi\ \text{ms}^{-1}$ Accept 9.42 ms^{-1}.	Correct $\frac{dx}{dt}$.	Correct answer with correct derivative.	

Q2	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$\frac{dy}{dx} = 3(\sin 2x)^2 \times (\cos 2x) \times 2$	Correct derivative.		
(b)	$N'(t) = -1 + \frac{2}{5}e^{\frac{t}{5}}$ $N'(10) = -1 + \frac{2}{5}e^{\frac{1}{5}\times 10}$ $= -1 + \frac{2}{5}e^2$ Accept 1.96.	Correct $N'(t)$.	Correct answer with correct derivative.	
(c)	(i) 1) 1, 2 2) $x < -2$ and $2 < x < 3$ 3) $x > 2$ (ii) Limit does not exist.	TWO out of four answers correct.	THREE out of four answers correct.	
(d)	$f'(x) = (-2x+6)e^{-x^2+6x-9}$ $= -2(x-3)e^{-x^2+6x-9}$ $f'(x) > 0$ $-2(x-3)e^{-x^2+6x-9} > 0$ $-2(x-3) > 0$ $x - 3 < 0$ $x < 3$ Therefore, f is increasing for $x < 3$.	Correct derivative $f'(x)$ AND Wrote down the inequality $f'(x) > 0$.	Correct range of x values.	

Q3	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$g'(x)=\dfrac{2\times(-\sin 2x)(x^2+1)-(\cos 2x)\times 2x}{\left(x^2+1\right)^2}$	Correct derivative.		
(b)	$2x^2+4xy=2a^2$ $4xy=2a^2-2x^2$ $y=\dfrac{2a^2-2x^2}{4x}$ $=\dfrac{a^2-x^2}{2x}$ $V=x^2y=x^2\times\dfrac{a^2-x^2}{2x}=\dfrac{a^2x-x^3}{2}$ $\dfrac{dV}{dx}=\dfrac{a^2-3x^2}{2}$ Solving $\dfrac{dV}{dx}=0$ gives $x=\dfrac{a}{\sqrt{3}}$. This is the x value that gives the maximum volume since $\dfrac{d^2V}{dx^2}=-3\times\dfrac{a}{\sqrt{3}}=-\sqrt{3}a<0$.	Correct volume function.	Correct x value with justification.	
(c)	$\dfrac{dy}{du}=2\left(1-\dfrac{1}{u^2}\right)=\dfrac{2(u^2-1)}{u^2}$ $\dfrac{dx}{du}=1+\dfrac{1}{u^2}=\dfrac{u^2+1}{u^2}$ $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$ $=2\left(\dfrac{u^2-1}{u^2}\right)\times\dfrac{u^2}{u^2+1}=2\dfrac{u^2-1}{u^2+1}$ $2\dfrac{u^2-1}{u^2+1}=\dfrac{1}{2}$ $4u^2-4=u^2+1$ $3u^2=5$ $u=\pm\sqrt{\dfrac{5}{3}}$ Accept ± 1.29	Correct derivatives $\dfrac{dx}{du}$ and $\dfrac{dy}{du}$.	Correct values.	

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Level 3 Calculus, 2020 v1

91579 Apply integration methods in solving problems

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply integration methods in solving problems.	Apply integration methods, using relational thinking, in solving problems.	Apply integration methods, using extended abstract thinking, in solving problems.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

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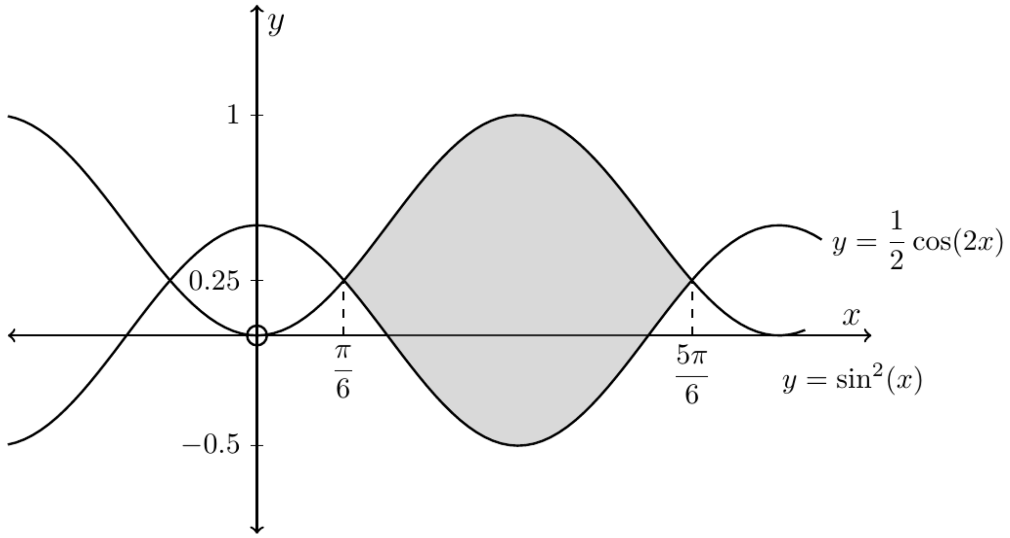
TOTAL

(d) The diagram below shows the graphs of the functions $y = \sin^2(x)$ and $y = \frac{1}{2}\cos(2x)$.



Find the area of the region shaded in the diagram.

You must use calculus and show the results of any integration needed to solve the problem.

QUESTION TWO

ASSESSOR'S USE ONLY

(a) Find $\int \cos(5x)\sin(3x)\ dx$.

(b) Use the values given below to find an approximation to $\int_2^{3.8} f(x)\ dx$ using the Trapezium rule.

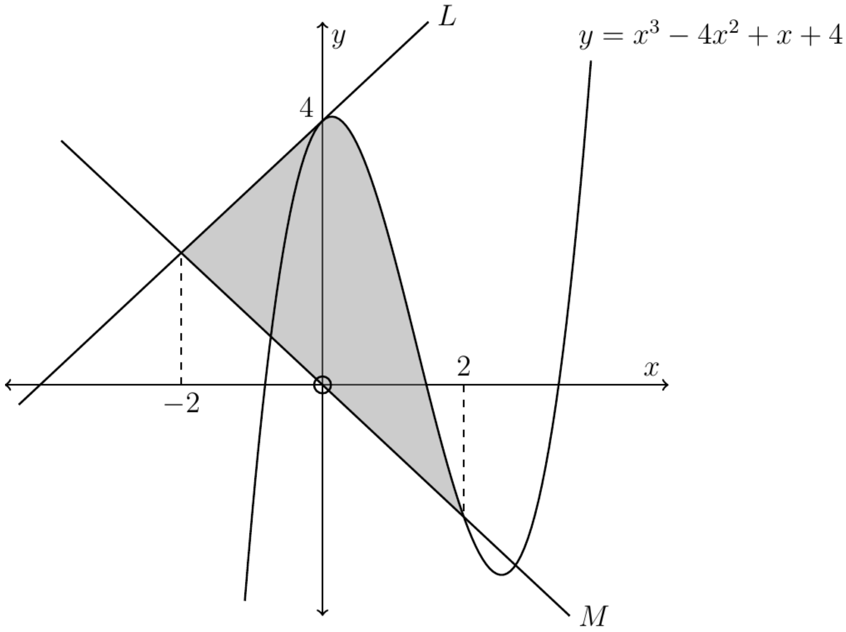
x	2	2.3	2.6	2.9	3.2	3.5	3.8
$f(x)$	3.3	3.8	4.4	5.1	6.0	7.0	8.2

(e) The diagram below shows the graph $y = x^3 - 4x^2 + x + 4$ and the lines L and M.

Line L is a tangent to the graph $y = x^3 - 4x^2 + x + 4$ at the point $(0, 4)$.

Line M passes through the origin and intersects line L at the point where $x = -2$.

Line M also intersects the graph $y = x^3 - 4x^2 + x + 4$ at the point where $x = 2$.



Find the area of the shaded region.

You must use calculus and give the results of any integration needed to solve this problem.

(c) Find the area enclosed between the graphs of the functions $y = x^2 - 1$ and $y = -x^2 + 2x + 3$.

You must use calculus and show the results of any integration needed to solve the problem.

ASSESSOR'S USE ONLY

Assessment Schedule – 2020 v1
Calculus: Apply integration methods in solving problems (91579)
Evidence Statement

Q1	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$\frac{1}{3}\times\frac{1}{2}\times(3x^2+5)^2+c$ Alternatively $\int 2x(3x^2+5)\ dx=\int(6x^3+10x)\ dx$ $=\frac{1}{4}\times 6x^4+\frac{1}{2}\times 10x^2+c$	Correct solution.		
(b)	$\frac{dy}{dx}=\frac{e}{x^2}$ $y=-\frac{e}{x}+c$ Since $y=e$ when $x=1$ $-e+c=e$ $c=2e$ Therefore, $y=-\frac{e}{x}+2e$.	Correct solution with correct integration.		
(c)	$\int_0^4 x\sqrt{x}\ dx=\int_0^4 x^{\frac{3}{2}}\ dx$ $=\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^4$ $=\frac{2}{5}(4)^{\frac{5}{2}}$ $=\frac{64}{5}$ Accept 12.8	Correct integration.	Correct solution with correct integration.	

Q1	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$\frac{dT}{dt}=\frac{30-T}{10}$ $\int \frac{1}{30-T}\,dT=\int \frac{1}{10}\,dt$ $-\ln\lvert 30-T\rvert=\frac{1}{10}t+c$ $\ln\lvert 30-T\rvert=-\frac{1}{10}t+c$ $\lvert 30-T\rvert=e^{-\frac{1}{10}t+c}$ $30-T=Ae^{-\frac{1}{10}t}$ $T=30-Ae^{-\frac{1}{10}t}$ Since $T=10$ when $t=0$, $A=20$. $T=30-20e^{-\frac{1}{10}t}$ Therefore $30-20e^{-\frac{1}{10}t}=28$ $e^{-\frac{1}{10}t}=0.1$ $-\frac{1}{10}t=\ln(0.1)$ $t=-10\times\ln(0.1)$ ≈ 23.026 After 23 minutes.	Correct integration.	Correct integration and particular solution of the DE.	Correct solution with correct integration.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	ONE correct derivative	2u	3u	1r	2r	1t with minor error(s).	1t

Q2	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$\int \cos(5x)\sin(3x)\ dx$ $= \frac{1}{2}\int \left(\sin(8x) - \sin(2x)\right)\ dx$ $= \frac{1}{2}\left(-\frac{1}{8}\cos(8x) + \frac{1}{2}\cos(2x)\right) + c$	Correct integration.		
(b)	$\int_2^{3.8} f(x)\ dx$ $\approx \frac{0.3}{2}\left(3.3 + 2(3.8 + 4.4 + 5.1 + 6.0 + 7.0) + 8.2\right)$ $= 9.615$	Correct answer.		
(c)	$\int_{-b}^{b} f(x)\ dx = -1.88 + 0.63 + 0.63 - 1.88$ $= -2.5$	Correct solution.		
(d)	By integration $v = -\frac{2}{t+1} + c$ Since $v = 2$ when $t = 0$ $-2 + c = 2$ $c = 4$ $v = -\frac{2}{t+1} + 4$ $s = \int_0^5 \left(-\frac{2}{t+1} + 4\right)\ dx$ $= \left[-2\ln\lvert t+1\rvert + 4t\right]_0^5$ $= \left(-2\ln(6) + 20\right) - \left(-2\ln(1) + 0\right)$ $= -2\ln(6) + 20$ ≈ 16.42 m The object travelled 16.42 m in the first 5 seconds.	Correct integration.	Correct integration and correct value of t.	